

Course «Algorithms» · Lecture 8

Dynamic Programming on Graphs: Warshall’s Algorithm

Textbook: A. Levitin, Section 8.4

Today’s question

In a directed graph, **who can reach whom?**

- the transitive closure of a digraph
- one key idea: which vertices may be stops in the middle
- Warshall’s algorithm in five lines, traced by hand
- next lecture: the same idea gives shortest paths

Questions about all pairs

- Does package A depend on package B, even through a chain?
- Can you fly from city A to city B on this network at all?
- Which web pages are linked by a chain of links?
- Do course prerequisites contain a cycle?

All of them are questions about **reachability**.

Adjacency matrix and transitive closure

Adjacency matrix A: A[i, j] = 1 if there is an edge i → j, else 0.

Transitive closure T:

$$T[i, j] = 1 \iff \text{there is a directed path of positive length from } i \text{ to } j$$

- T[i, i] = 1 exactly when i lies on a **cycle**
- Obvious way: DFS from every vertex, Θ(n(n + e))

Key idea: who may be in the middle?

Number the vertices 1, ..., n. Restrict only the **intermediate** vertices of a path:

$$R^{(k)}[i,j] = 1 \iff \text{a path } i \to j \text{ has all intermediate vertices in } \{1,\ldots,k\}$$

$$R^{(0)} = A, \qquad R^{(n)} = T$$

$$R^{(0)},\ R^{(1)},\ \ldots,\ R^{(n)}$$

Dynamic programming over one parameter k; each table entry is a whole matrix.

Warshall’s recurrence

Does the path use vertex k? We may assume at most once: a second visit means a cycle we can cut out.

No: it was there before

$$R^{(k-1)}[i, j] = 1$$

Yes: i reaches k, k reaches j

$$R^{(k-1)}[i, k] = 1 \quad \text{and} \quad R^{(k-1)}[k, j] = 1$$

$$R^{(k)}[i, j] = R^{(k-1)}[i, j] \ \vee \ \left(R^{(k-1)}[i, k] \ \wedge \ R^{(k-1)}[k, j]\right)$$

The rule for working by hand

At stage k, every row with a 1 in column k absorbs row k.

R[i, k] = 1 means “i reaches k”, so i also reaches everything k reaches.

The algorithm

```
Warshall(A[1..n, 1..n])
  R ← A
  for k ← 1 to n do
    for i ← 1 to n do
      for j ← 1 to n do
        R[i, j] ← R[i, j] or (R[i, k] and R[k, j])
  return R
```

The loop over **k** must be the outermost one.

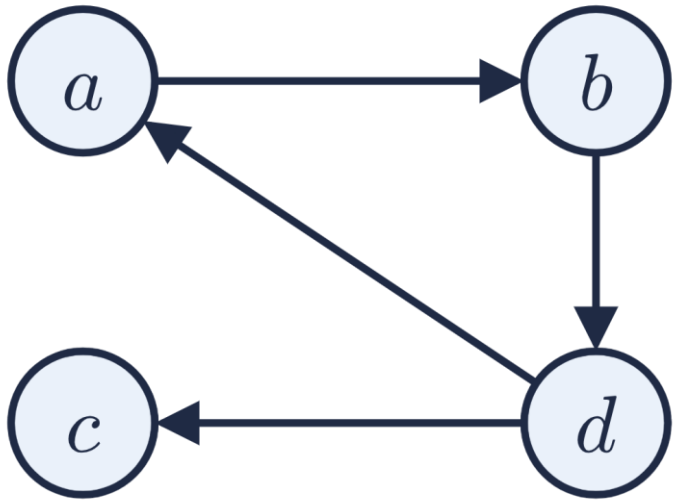
Why can we overwrite one matrix?

The recurrence needs stage k – 1 values. But during stage k, **row k and column k never change**:

$$R[k, j] \leftarrow R[k, j] \vee (R[k, k] \wedge R[k, j]) = R[k, j]$$

So R[i, k] and R[k, j] still hold the values from stage k – 1.

Example: the graph and $R^{(0)} = A$



	a	b	c	d
a	0	1	0	0
b	0	0	0	1
c	0	0	0	0
d	1	0	1	0

Stages k = a and k = b

$R^{(1)}$: d absorbs row a

	a	b	c	d
a	0	1	0	0
b	0	0	0	1
c	0	0	0	0
d	1	1	1	0

d → a → b, so now d reaches b.

$R^{(2)}$: a and d absorb row b

	a	b	c	d
a	0	1	0	1
b	0	0	0	1
c	0	0	0	0
d	1	1	1	1

R[d, d] = 1: the cycle d → a → b → d.

Stages k = c and k = d

k = c: nothing changes

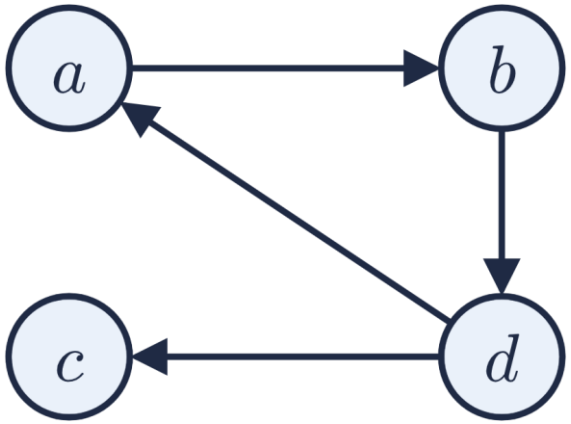
Column c has a 1 in row d, but row c is all zeros: c reaches nothing.

$R^{(3)} = R^{(2)}$

$R^{(4)}$ = T: rows a, b, d absorb row d

	a	b	c	d
a	1	1	1	1
b	1	1	1	1
c	0	0	0	0
d	1	1	1	1

Does it make sense?



- a → b → d → a is a cycle: a, b, d reach each other and themselves
- c is a dead end: its row is all zeros
- The diagonal is 1 exactly on the cycle

How fast is it?

$$\text{time} = \Theta(n^3), \qquad \text{space} = \Theta(n^2)$$

Bit trick: OR whole rows at once (w = word size, e.g. 64)

$$\text{if } R[i,k] = 1: \quad \text{row}_i \leftarrow \text{row}_i \vee \text{row}_k \qquad \Rightarrow \qquad \Theta(n^3/w)$$

Method	Time	Better for
Warshall	Θ(n³), Θ(n³/w)	dense graphs, matrix input
DFS from each vertex	Θ(n(n + e))	sparse graphs, lists

Common mistakes

- **Wrong loop order:** k must be the outermost loop
- **Mixed conventions:** fix once whether A[i, j] means i → j or j → i
- **The diagonal:** here R[i, i] = 1 only on a cycle; some books always put 1

Summary

- Transitive closure: who reaches whom; T[i, i] = 1 on a cycle
- DP over allowed intermediate vertices, from $R^{(0)}$ = A to $R^{(n)}$ = T
- By hand: rows with a 1 in column k absorb row k
- One matrix in place; time **Θ(n³)**, space **Θ(n²)**

$$R^{(k)}[i, j] = R^{(k-1)}[i, j] \ \vee \ \left(R^{(k-1)}[i, k] \ \wedge \ R^{(k-1)}[k, j]\right)$$

Review questions

1. Define the transitive closure. What does a 1 on the diagonal mean?
2. What does $R^{(k)}[i, j]$ denote? Derive Warshall's recurrence. Why is it enough to use vertex k at most once?
3. Compute the closure for edges $1 \to 2$, $2 \to 3$, $3 \to 1$, $3 \to 4$. Show all intermediate matrices.